

Linear Algebra I

10/11/2023, Friday, 15:00 – 17:00

1 Diagonalization

3 + 3 + 3 + 3 + 3 + 10 = 25 pts

Consider the matrix

$$M = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

- (a) Without computing eigenvalues of M , find the sum and product of them.
- (b) Without computing eigenvalues of M , determine whether it is diagonalizable. (Justify your answer!)
- (c) Find its characteristic polynomial.
- (d) Show that -2 is an eigenvalue of M .
- (e) Find all eigenvalues of M .
- (f) Is M unitarily diagonalizable? If yes, find an orthogonal diagonalizer.

REQUIRED KNOWLEDGE: Sum/product of eigenvalues and their relation to trace and determinant, (unitarily) diagonalizability, characteristic polynomial, eigenvalues/vectors, finding diagonalizers.

SOLUTION:

1a: The sum of the eigenvalues is equal to the trace and the product to the determinant. Therefore, we see that the sum of eigenvalues is 0 whereas the product is equal to

$$\det \left(\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \right) = 0 - 1 - 1 - 0 - 0 - 0 = -2$$

by using Sarrus' rule.

1b: We know that Hermitian matrices are diagonalizable. Since M is real and symmetric, it is Hermitian. Hence, we can conclude that it is diagonalizable.

1c: The characteristic polynomial of M is given by $p_M(\lambda) = \det(\lambda I - M)$. Note that

$$\begin{aligned} \det(\lambda I - M) &= \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = \lambda^3 + 1 + 1 - \lambda - \lambda - \lambda \\ &= \lambda^3 - 3\lambda + 2. \end{aligned}$$

1d: Note that $p_M(-2) = (-2)^3 - 3(-2) + 2 = 0$. Therefore, -2 is a root of the characteristic polynomial and hence is an eigenvalue.

1e: From the previous, we know that -2 is an eigenvalue. As such, $p_M(\lambda) = (\lambda + 2)q(\lambda)$ where q is a monic polynomial of degree 2 which can be obtained by dividing $\lambda^3 - 3\lambda + 2$ by $\lambda + 2$:

$$\frac{\lambda^3 - 3\lambda + 2}{\lambda + 2} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

Therefore, we can conclude that $p_M(\lambda) = (\lambda + 2)(\lambda - 1)^2$. Consequently, the eigenvalues of M are $\lambda_1 = -2$ and $\lambda_2 = 1$.

1f: Note that M being real and symmetric. Since Hermitian matrices are unitarily diagonalizable, so is M . To find an orthogonal diagonalizer, we proceed with finding eigenvectors of M .

For $\lambda_1 = -2$, we have

$$\lambda_1 I - M = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

As such, eigenvectors corresponding to the eigenvalue -2 are nontrivial solutions of

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0}_3. \quad (1)$$

To solve this homogenous system of linear equations (1), we proceed with putting its coefficient matrix into the reduced row echelon form:

$$\begin{aligned} & \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{\textcircled{1} \leftrightarrow \textcircled{2}} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\ & \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{\begin{matrix} \textcircled{2} \leftarrow \textcircled{2} + 2 \cdot \textcircled{1} \\ \textcircled{3} \leftarrow \textcircled{3} - 1 \cdot \textcircled{1} \end{matrix}} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \\ & \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{\begin{matrix} \textcircled{3} \leftarrow \textcircled{3} + 1 \cdot \textcircled{2} \\ \textcircled{3} \leftarrow \textcircled{3} - 1 \cdot \textcircled{1} \end{matrix}} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\textcircled{3} \leftarrow -\frac{1}{3} \cdot \textcircled{3}} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\textcircled{1} \leftarrow \textcircled{1} + 2 \cdot \textcircled{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Now, we see that the third unknown is a free variable whereas the first two are lead variables. This gives us the general solution:

$$\begin{bmatrix} a \\ a \\ a \end{bmatrix}$$

where a is a scalar. By taking $a = 1$, we see that

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda_1 = -2$.

For $\lambda_2 = 1$, we have

$$\lambda_2 I - M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

As such, eigenvectors corresponding to the eigenvalue -2 are nontrivial solutions of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}_3.$$

To solve this homogenous system of linear equations (1), we proceed with putting its coefficient matrix into the reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{\textcircled{2} \leftarrow \textcircled{2} - 1 \cdot \textcircled{1} \\ \textcircled{3} \leftarrow \textcircled{3} - 1 \cdot \textcircled{1}}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, we see that the last two unknowns are free variables whereas the first is a lead variable. This gives us the general solution:

$$\begin{bmatrix} -a - b \\ a \\ b \end{bmatrix}$$

where a, b are scalars.

By taking $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$, we see that

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent eigenvectors for $\lambda_2 = 1$.

To obtain an orthogonal diagonalizer, we need to orthonormalize eigenvectors.

For $\lambda_1 = -2$, we take

$$\mathbf{y}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

since $\|\mathbf{x}_1\| = \sqrt{\mathbf{x}_1^T \mathbf{x}_1} = \sqrt{3}$.

For $\lambda_1 = -2$, we take

$$\mathbf{y}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

since $\|\mathbf{x}_2\| = \sqrt{\mathbf{x}_2^T \mathbf{x}_2} = \sqrt{2}$. By employing Gram-Schmidt process, we let

$$\mathbf{z}_3 = \mathbf{x}_3 - (\mathbf{y}_2^T \mathbf{x}_3) \mathbf{y}_2.$$

Since $\mathbf{y}_2^T \mathbf{x}_3 = \frac{1}{\sqrt{2}}$, we see that

$$\mathbf{z}_3 = \mathbf{x}_3 - \frac{1}{\sqrt{2}} \mathbf{y}_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Note that $\|\mathbf{z}_3\|^2 = \mathbf{z}_3^T \mathbf{z}_3 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$. Hence, we obtain

$$\mathbf{y}_3 = \frac{\mathbf{z}_3}{\|\mathbf{z}_3\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

Therefore, the orthogonal matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

is a diagonalizer. Indeed, one can verify that

$$\begin{aligned} MU &= \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \\ U \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}. \end{aligned}$$

Therefore, we see that

$$M = U \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T.$$

(a) Let $a, b \in \mathbb{R}$ and

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid (x_1 - ax_2)(x_1 - bx_2) = 0\}.$$

Determine all values of a and b such that S is a subspace of \mathbb{R}^2 .

(b) Let

$$S_1 = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad S_2 = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

Find bases for the subspaces:

(i) $S_1 \cap S_2$

(ii) $S_1 + S_2$

REQUIRED KNOWLEDGE: **Subspaces and bases for subspaces.**

SOLUTION:

2a: Note that $\mathbf{0}_2 \in S$ for all values of a and b . Also, the set S is closed under scalar multiplication for all values of a and b . To see this, let $\mathbf{x} \in S$ and α be a scalar. Then, we have $(\alpha x_1 - a\alpha x_2)(\alpha x_1 - b\alpha x_2) = \alpha^2(x_1 - ax_2)(x_1 - bx_2) = 0$. Hence, $\alpha\mathbf{x} \in S$.

Suppose, now, that S is a subspace, that is, it is closed under vector addition. Note that $\begin{bmatrix} a \\ 1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \end{bmatrix} \in S$. Therefore, $\begin{bmatrix} a+b \\ 2 \end{bmatrix} \in S$. This means that $(a+b-2a)(a+b-2b) = 0$. Since the right hand side is equal to $-(a-b)^2$, this can happen only if $a = b$. This proves that if S is a subspace then $a = b$.

Suppose, now, that $a = b$. Then, we have $S = \{\mathbf{x} \in \mathbb{R}^2 \mid (x_1 - ax_2)(x_1 - ax_2) = 0\} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = ax_2\}$. We already know that this set is nonempty and closed under scalar multiplication. Let $\mathbf{x}, \mathbf{y} \in S$. Note that $x_1 + y_1 = ax_2 + ay_2 = a(x_2 + y_2)$. This proves that $\mathbf{x} + \mathbf{y} \in S$ and hence S is a subspace.

Therefore, we have proven that S is a subspace if and only if $a = b$.

2b(i): To determine a basis for $S_1 \cap S_2$, we first observe that

$$\mathbf{x} \in S_1 \text{ if and only if } \mathbf{x} = \begin{bmatrix} a \\ a \\ a \\ b \end{bmatrix} \tag{2}$$

for some scalars a, b and

$$\mathbf{x} \in S_2 \text{ if and only if } \mathbf{x} = \begin{bmatrix} c \\ c \\ d \\ d \end{bmatrix} \tag{3}$$

for some scalars c, d . Therefore, $\mathbf{x} \in S_1 \cap S_2$ if and only if

$$\mathbf{x} = \begin{bmatrix} a \\ a \\ a \\ b \end{bmatrix} = \begin{bmatrix} c \\ c \\ d \\ d \end{bmatrix}$$

for some scalars a, b, c, d . From the last equality, we see that $a = c$, $a = d$, and $b = d$. This means that $a = b = c = d$. Therefore, we can conclude that

$$\mathbf{x} \in S_1 \cap S_2 \text{ if and only if } \mathbf{x} = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}$$

for some scalar a . Consequently, the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

forms a basis for $S_1 \cap S_2$.

2b(ii): To determine a basis for $S_1 + S_2$, we first observe from (2) and (3) that

$$\mathbf{x} \in S_1 + S_2 \text{ if and only if } \mathbf{x} = \begin{bmatrix} a+c \\ a+c \\ a+d \\ b+d \end{bmatrix} = (a+c) \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (a+d) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (b+d) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (4)$$

for some scalars a, b, c, d . Then, we see that

$$S_1 + S_2 \subseteq \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

The reverse inclusion holds as well. This can be seen in two different ways. One option is to invoke the fact that

$$\dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$$

which was proven in the last homework assignment. Since $\dim(S_1) = \dim(S_2)$ (follows readily from the way S_1 and S_2 given) and $\dim(S_1 \cap S_2) = 1$ (shown in 2b(i)), we see that $\dim(S_1 + S_2) = 2 + 2 - 1 = 3$. Therefore, we can conclude from (4) and the linear independence of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

that

$$S_1 + S_2 = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right). \quad (5)$$

As an alternative option, we can prove directly that the reverse inclusion holds:

Let

$$\mathbf{x} \in \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Then,

$$\mathbf{x} = \begin{bmatrix} e \\ e \\ f \\ g \end{bmatrix}$$

for some scalars e, f, g . Note that

$$\mathbf{x} = e \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (g - f + e) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (f - e) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Since the first two vectors in the expression above belong to S_1 and the last belongs to S_2 , we see that $\mathbf{x} \in S_1 + S_2$.

Since the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent, we can conclude from (5) that they form a basis for $S_1 + S_2$.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- (a) Find the eigenvalues of A .
- (b) Is A diagonalizable? Why?
- (c) Find the Jordan canonical form of A .

REQUIRED KNOWLEDGE: Unitary diagonalization.

SOLUTION:

3a: The matrix A is a triangular matrix. Therefore, its eigenvalues are its diagonal entries. So, A has only 1 as an eigenvalue.

3b: To determine whether A is diagonalizable, we need to check whether it has 4 linearly independent eigenvectors, or equivalently $\dim N(I - A) = 4$. To do so, we proceed by solving the following homogenous system linear equation:

$$\mathbf{0} = (I - A)\mathbf{x} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

One can write down the solution of this system right away:

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$$

where a, b are scalars. Then, we see that $\dim N(I - A) = 2$ and hence A is not diagonalizable.

3c: To find out the Jordan canonical form, we need to compute the Weyr characteristics of A . To do so, we proceed with computing the ranks of the powers of $(I - A)$:

$$\begin{aligned} \text{rank}(I - A)^0 &= \text{rank } I_4 = 4 \\ \text{rank}(I - A)^1 &= \text{rank} \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2 \\ \text{rank}(I - A)^2 &= \text{rank} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 1 \\ \text{rank}(I - A)^3 &= \text{rank} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0. \end{aligned}$$

From the last equation, we see that $\text{rank}(I - A)^k = 0$ for all $k \geq 3$. Therefore, the Weyr characteristics of A are

$$\begin{aligned}w_1 &= 4 - 2 = 2 \\w_2 &= 2 - 1 = 1 \\w_3 &= 1 - 0 = 1 \\w_4 &= 0 - 0 = 0.\end{aligned}$$

This means the index $\kappa = 3$. Now, we compute

$$\begin{aligned}\rho_1 &= w_1 - w_2 = 1 \\ \rho_2 &= w_2 - w_3 = 0 \\ \rho_3 &= w_3 - w_4 = 1.\end{aligned}$$

Consequently, the Jordan canonical form of A consists of 1 Jordan block of size 1×1 and 1 Jordan block of size 3×3 . Therefore, A is similar to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & \\ \hline & & & 1 \end{array} \right].$$

Let a, b, c, d be scalars and $M \in \mathbb{F}^{m \times m}$. Consider the matrices

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} aM & bM \\ cM & dM \end{bmatrix}.$$

Show that $\mu\lambda$ is an eigenvalue of N if μ is an eigenvalue of K and λ is an eigenvalue of M .

(HINT: Let (μ, \mathbf{x}) be an eigenpair of K and (λ, \mathbf{z}) be an eigenpair of M . Construct an eigenvector of N corresponding to the eigenvalue $\mu\lambda$ in terms of \mathbf{x} and \mathbf{z} .)

REQUIRED KNOWLEDGE: Eigenvalues/eigenvectors.

SOLUTION:

4: Let (μ, \mathbf{x}) be an eigenpair of K and (λ, \mathbf{z}) be an eigenpair of M . Then, we have

$$\begin{bmatrix} \mu x_1 \\ \mu x_2 \end{bmatrix} = \mu \mathbf{x} = K \mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \quad (6)$$

and

$$M \mathbf{z} = \lambda \mathbf{z}. \quad (7)$$

Note that

$$N \begin{bmatrix} x_1 \mathbf{z} \\ x_2 \mathbf{z} \end{bmatrix} = \begin{bmatrix} aM & bM \\ cM & dM \end{bmatrix} \begin{bmatrix} x_1 \mathbf{z} \\ x_2 \mathbf{z} \end{bmatrix} = \begin{bmatrix} (ax_1 + bx_2)M \mathbf{z} \\ (cx_1 + dx_2)M \mathbf{z} \end{bmatrix} \stackrel{(6) \text{ and } (7)}{=} \mu\lambda \begin{bmatrix} x_1 \mathbf{z} \\ x_2 \mathbf{z} \end{bmatrix}$$

Since both \mathbf{x} and \mathbf{z} are nonzero, the vector $\begin{bmatrix} x_1 \mathbf{z} \\ x_2 \mathbf{z} \end{bmatrix}$ is nonzero. Therefore, we see that $\mu\lambda$ is an eigenvalue of N .
